Appendix C

Derivation of the ratio distance function

Our objective is to derive a function to measure the distance between two pairs of RGB ratios. The ratio between two RGB values p_i, p_j is denoted

$$r_1 = \left(r_R^1, r_G^1, r_B^1\right) = \left(\frac{R_i^1}{R_j^1}, \frac{G_i^1}{G_j^1}, \frac{B_i^1}{B_j^1}\right)$$

Also consider another colour ratio $r_2 = \left(r_R^2, r_G^2, r_B^2\right)$ between two other colours q_i and q_j. To compute the distance between r_1 and r_2 the average of the three band-wise distances is calculated as the arithmetic average of the channel-wise differences:

$$d_{rat}(r_1, r_2) = \frac{1}{3}\left(d_{fr}(r_R^1, r_R^2) + d_{fr}(r_G^1, r_G^2) + d_{fr}(r_B^1, r_B^2)\right) \tag{C.1}$$

We seek a function for the distance d_{fr} between two 1-dimensional ratios. Let $p = \frac{a}{b}$ and $q = \frac{c}{d}$ be 1-dimensional ratios where $a, b, c, d \in \mathbb{R}^+$. Here, a, b, c, d correspond to R, G, B band values and therefore $a, b, c, d \in [0, 255]$. The distance between p and q is denoted as

$$d_{fr}(p, q) = d_{fr}(\frac{a}{b}, \frac{c}{d}) \tag{C.2}$$

In a different image of the same scene, the original colours a, b, c, d are likely to change as a result of uniformly distributed additive noise. The new ratios p' and q' are $p' = \frac{a+\epsilon_a}{b+\epsilon_b}$ and $q = \frac{c+\epsilon_b}{d+\epsilon_d}$ where $|\epsilon_i| < \Delta, i = a, b, c, d$ and $\Delta \in \mathbb{R}^+$ and the distance between them is

$$d_{fr}(p', q') = d_{fr}(\frac{a + \epsilon_a}{b + \epsilon_b}, \frac{c + \epsilon_b}{d + \epsilon_d}) \tag{C.3}$$

We seek a distance function d_{fr} between a pair of 1-dimensional ratios p and q with the following properties:

(a) $d_{fr}(p, p) = 0$ for any ratio p.

(b) The distance is symmetric i.e. $d_{fr}(p, q) = d_{fr}(q, p)$ since the order of the RGB ratios in each pair is not fixed

(c) If the ratios p, q and $p', q'$are computed from the same pair of surfaces viewed in two different images under the same imaging conditions then the distance between them should be a function of only the noise.

Such a function for 1-dimensional ratios p and q is:

$$d_{fr}(p, q) = \frac{|ad - bc|}{a + b + c + d} \tag{C.4}$$

Eq C.4 satisfies properties (a) and (b). Proof of property (c) is shown below. We want to prove that

$$d_{fr}(p', q') - d_{fr}(p, q) = f(\Delta, p, q, p', q') = g(\Delta) \tag{C.5}$$

Since $|e_i| \leq \Delta$, the maximum distance f is observed when $\epsilon_a = \epsilon_d = \Delta$ and $\epsilon_b = \epsilon_c = -\Delta$. We will show that using Eq. C.4, the function f is indeed a function of Δ.

$$d_{fr}(p', q') =$$

$$d_{fr}(\frac{a + \Delta}{b - \Delta} \frac{c - \Delta}{d + \Delta}) =$$

$$\frac{|ad + a\Delta + d\Delta + \Delta^2 - bc + b\Delta + c\Delta - \Delta^2|}{a + \Delta + b - \Delta + c - \Delta + d + \Delta|} =$$

$$\frac{ad - bc + \Delta(a + b + c + d)}{a + b + c + d} =$$
$$\frac{ad - bc}{a + b + c + d} + \Delta =$$
$$d_{fr}(p, q) + \Delta$$

Therefore,

$$f(\Delta, a, b, c, d) = d_{fr}(p', q') - d_{fr}(p, q)) = \Delta = g(\Delta)$$